We now show to use the extension problem to recover the original group.

We have spectral sequences:

$$E^2_{a,b} = H_a(G; \pi_b(E)) \text{ (htpy orbit)} \Longrightarrow \pi_{a+b}(E_{hG})$$
$$E^2_{a,b} = H^{-a}(G; \pi_b(E)) \text{ (htpy fixed point)} \Longrightarrow \pi_{a+b}(E^{hG})$$
$$E^2_{a,b} = \hat{H}^{-a}(G; \pi_b(E)) \text{ (tate)} \Longrightarrow \pi_{a+b}(E^{tG})$$

Where $\hat{H}^*(G; M)$ is the tate cohomology of $G$-module $M$, defined by:

$$\hat{H}^n(G; M) := \begin{cases} H^n(G; M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M) & \text{if } n \geq 1 \\ \mathrm{coker}(\mathrm{Norm}_G) & \text{if } n = 0 \\ \ker(\mathrm{Norm}_G) & \text{if } n = -1 \\ H_{-n-1}(G; M) = \mathrm{Tor}^{\mathbb{Z}[G]}_{-n-1}(\mathbb{Z}, M) & \text{if } n \leq -2 \end{cases}$$

Let $E = \mathrm{KU}^\wedge_p, G = C_p$ , choose the $G$-action on $\mathrm{KU}^\wedge_p$ to be the trivial action.

Recall that the homotopy group of $\mathrm{KU}^\wedge_p$ is $\mathbb{Z}^\wedge_p[u, u^{-1}]$, with $\deg(u) = 2$. We get the $E_2$ page of the tate spectral sequence:

$$E^2_{a,2b'} = \begin{cases} \mathrm{Ext}^{-a}_{\mathbb{Z}[G]}\left(\mathbb{Z}, \mathbb{Z}^\wedge_p\right) = \mathbb{Z}/p\mathbb{Z} & \text{if } a = -2(n+1) \\ \ker\left(\mathbb{Z}^\wedge_p \xrightarrow{\mathrm{Norm}} \mathbb{Z}^\wedge_p\right) = 0 & \text{if} = -1 \\ \mathrm{coker}\left(\mathbb{Z}^\wedge_p \xrightarrow{\mathrm{Norm}} \mathbb{Z}^\wedge_p\right) = \mathbb{Z}/p\mathbb{Z} & \text{if } a = 0 \\ \mathrm{Tor}^{\mathbb{Z}[G]}_a\left(\mathbb{Z}, \mathbb{Z}^\wedge_p\right) = \mathbb{Z}/p\mathbb{Z} & \text{if } a = 2(n+1) \end{cases}$$

By using the following projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[C_p]$-module:

$$\cdots \longrightarrow \mathbb{Z}[C_p] \xrightarrow{\mathrm{Norm}} \mathbb{Z}[C_p] \xrightarrow{e-\alpha} \mathbb{Z}[C_p] \xrightarrow{\mathrm{Norm}} \mathbb{Z}[C_p] \xrightarrow{e-\alpha} \mathbb{Z}[C_p] \xrightarrow{\mathrm{aug}} \mathbb{Z} \longrightarrow 0$$

We picture the $E_2$-page, each dot present an additive group $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

Now we can observe that, for degree reason, there are no non-trivial differentials in all $E_{n\geq 2}$ pages. Thus, the $E_\infty$-page is just the $E_2$-page, to recover the original group, we need additional knowledge about the multiplication to solve the extension problem and recover the ring structure.

We know that ${\mathrm{KU}^\wedge_p}^{tG} = \mathrm{cofiber}\left({\mathrm{KU}^\wedge_p}_{hG} \xrightarrow{\mathrm{Norm}} {\mathrm{KU}^\wedge_p}^{hG}\right)$, thus ${\mathrm{KU}^\wedge_p}^{tG}$ is a $\mathbb{E}_\infty$-module over ${\mathrm{KU}^\wedge_p}^{hG}$, and the multiplication on ${\mathrm{KU}^\wedge_p}^{tG}$ is compatible with ${\mathrm{KU}^\wedge_p}^{tG}$-multiplication.

Using the following Gysin-type sequence:

$$S^1 \longrightarrow \mathrm{B}C_p \longrightarrow \mathrm{B}S^1$$

And combine with the complete multiplication ... We get the following result: $\pi_* {\mathrm{KU}^\wedge_p}^{hC_p} = \left(\pi_* \mathrm{KU}^\wedge_p\right)[\![x]\!]/([p]x)$ with $\deg(x) = -2$, where $[p]x$ is $\underbrace{x +_F \cdots +_F x}_{p \text{ times}}$ ($+_F$ presents the formal group addition).

Use Araki's formula [1](A2.2.4), we get: $\pi_* {\mathrm{KU}^\wedge_p}^{hC_p} = \mathbb{Z}^\wedge_p[u, u^{-1}][\![x]\!]/(px - u^{p-1}x^p)$

Similarly we can picture the $E_2$ page of the homotopy fixed point spectral sequence:

We can see the only multiplicate by $p$-extension problem is: $p \times x = u^{p-1}x$, using the compatibility of multiplications on $\mathrm{KU}_p^{\wedge\, tC_p}$, we claim that this is the only multiplicate by $p$-extension problem in the $E_\infty$-page of tate spectral sequence for $\pi_*\mathrm{KU}_p^{\wedge\, tC_p}$. Using this way, we claim that:

$$\begin{aligned}
\pi_*\mathrm{KU}_p^{\wedge\, tC_p} &= \mathbb{Z}_p^\wedge[u, u^{-1}][\![x]\!][x^{-1}]/(px - u^{p-1}x^p) \\
&= \mathbb{Z}_p^\wedge[ux, (ux)^{-1}][\![x]\!]/(p - u^{p-1}x^{p-1}) \\
&= \mathbb{Z}_p^\wedge\left[\sqrt[p-1]{-p}, \left(\sqrt[p-1]{-p}\right)^{-1}\right][\![x]\!] \\
&= \mathbb{Q}_p^\wedge(\zeta_p)(\!(x)\!)
\end{aligned}$$